Grade 9/10 Math Circles

February 15

The Shape of You

Introduction

Let's recall what we've done so far. By considering these shapes called n-simplexes – dots, lines, triangles, tetrahedrons, and so on – and counting the number of faces they had of each dimension, we noticed a cool pattern, and recorded our results in a chart that looked very much exactly like Pascal's Triangle...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

This let us make predictions about what a hypothetical "4-dimensional" and "5-dimensional" tetrahedron would look like. If you don't remember or missed the class, then for our purposes we can just define an n**-simplex** to be the shape you get by drawing n dots and joining every possible pair of dots together with a line (and then filling in the interior of that whole space created by those lines and dots). A **face** is just a "piece" of that shape, like a dot, a line, a triangle, etc.

How did we count faces of a particular dimension? We defined the **dimension** of a dot, line, triangle, etc. to be the number of dots, minus one.

Stop and Think

For which shapes does this definition make sense? [a]

[a] Personally, I'm of the opinion that saying a square has dimension $4 - 1 = 3$ is absolute bollocks.

So say we're counting the number of faces that have dimension 6 in a 10-simplex. We must be looking for faces that have 7 dots. Since every dot is connected to every other dot, any and all sets of 7 dots that we choose out of the total 10 will give a totally valid face of dimension $7 - 1 = 6$. So what's the number of ways to choose 7 dots out of 10?

For this, we define the following symbol:

$$\binom{n}{k} = \text{The number of ways to choose } k \text{ dots out of a total of } n \text{ dots}$$

So the answer for our example should be $\binom{10}{7}$. Obviously, this is useless. We still don't know how to compute this number, and giving this weird computation a weird symbol doesn't really justify a pure mathematics research grant, does it?

Luckily! We noticed a cool pattern in Pascal's Triangle: namely, every number is the sum of the previous two numbers above it. See the Session 1 notes for details on why this is [1].

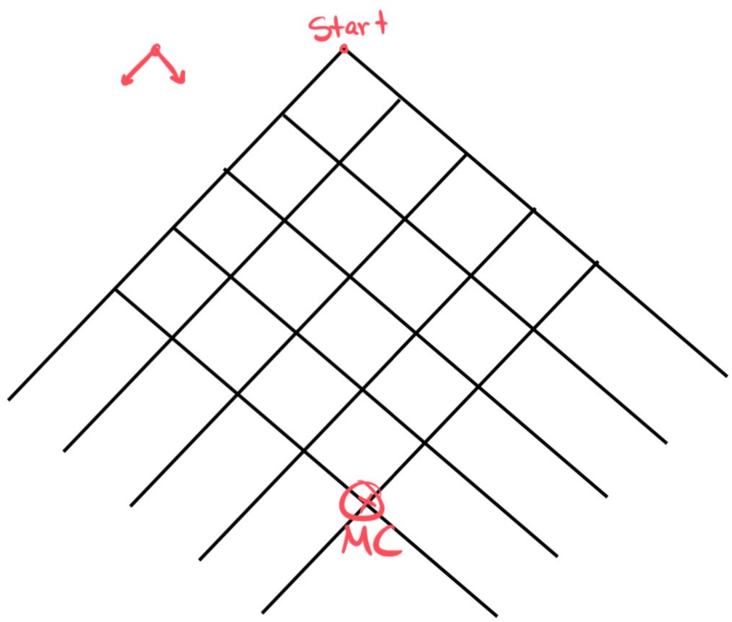
For practice, in the space below, give an argument for why this "addition rule" in Pascal's Triangle is true:

Let's see what else we can do with these ideas!

[1]Frankly, my voice is hoarse and my typing fingers are sore from doing this proof so much. Please be nice to me :)

Paths (to a Mathematician's Heart)

At the end of last class, we started path-counting problems. It goes like this: suppose I need to get to MC as fast as possible. The map of Waterloo looks like this (not to scale):



So every move I make has to be in the downwards direction, but at each intersection, I can choose to go left or right.

Warm-Up

How many ways are there to get to MC? More generally, how many ways are there to get to any intersection? Write your answers on the intersections in the grid above. We'll need this later!

Pair up with a friend[a] and try to explain *why* your answers are what they are.

[a]an enemy is an acceptable alternative

More details on this can be found in the Session 1 Problem Set.

Polynomial (Dreamin')

Let's do something totally unrelated now because transitions are hard and you try coming up with math-related content after eating a 400g bag of no name™ marshmallows.

Polynomials are cool! **Polynomials** are just expressions where you add (and subtract) multiples of terms with variables of different powers. For example:

$$4x^2 + 3x - 7$$

$$ab^2 + 9b + 378a^4b^2c + 3$$

$$6$$

$$0$$

$$y$$

$$z + y$$

$$2zy$$

Are all examples of polynomials[2]. Notice:

$$2^x$$

is NOT a polynomial because the variable is in the *exponent*, meaning this function has a different behaviour from the other ones above, and this difference matters enough for us to exclude it right now.

Arithmetic

We can do airthmetic with polynomials just like regular numbers! We call the "pieces" of a polynomial its **terms**, and two **terms** can be added and subtracted if the variables are all mutliplied and raised to powers in the same way (otherwise, let's just not do anything). For example:

$$x + y + 3y = x + 4y$$

$$ab + ba = 2ab$$

[2]Got it? Good.

$$7 - ab^2 = 7 - ab^2$$

What about multiplication? How do we multiply polynomials together? Can we even multiply them? Beats me, but let's figure it out!

Let's try to figure out how to multiply polynomials together. Right now, we only have a definition of multiplication that works for regular numbers. But what if we were really stubborn and wanted to multiply *polynomials*[a] too? Use your knowledge of how regular multiplication works to extend the definition of multiplication to polynomials in a sensible way:

$$\begin{array}{r} 2\ 0\ 7\ 3 \\ \times \quad 4\ 6\ 1 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 2x + 7y + 3xy^2 \\ \times \quad 4x^2 - 6y + xy^2 \\ \hline \end{array}$$

[a] Note that every number can be viewed as a specific type of polynomial

Bonus Bonus Bonus
You know I'm going to ask this too... what about division?

$$12 \overline{)\ 6\ 9\ 6} \qquad\qquad x + 2 \overline{)\ 6x^2 + 9x + 6}$$

Crossover Event

Out of curiousity, try the following computations:

Factored	Expanded
$(a+b)$	
$(a+b)^2$	
$(a+b)^3$	
$(a+b)^4$	
$(a+b)^5$	

Notice any patterns? Can you explain why? The space below is left blank for rough work:

To Dimension-3, and Beyond!

So, I love Pascal's Triangle, but sometimes it can get a bit... *two-dimensional*[3]. So let's see what a three dimensional Pascal's Triangle might look like!

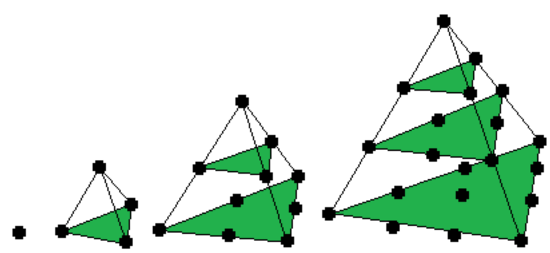
[3]Was that the sound of a sheep, a drum, and a snake falling off a cliff? Because I just heard *ba-dum-tsss*

Warning: The following activity involves a risk of **not working** if the mathematicians involved are avid sugar-enjoyers and hungry. Please take all precautions necessary.

In groups of four or five (or at home), using *marshmallows* and *toothpicks*, construct a three dimensional "tetrahedral grid". This will take some thinking and planning! Each group will only need a few layers of the grid, but we may attempt to put all these grids together to construct ONE GIANT TETRAHEDRON.

The group with the largest tetrahedron will split the remainder of my marshmallows. Note: each person will get a limited number of marshmallows. Hence, social competition will (hopefully) prevent too much unlawful consumption of mathematically-purposed marshmallows.

Now we might have something that looks like this:



If we assume that this will have some cool numbers and patterns associated with this, just like Pascal's Triangle, then a good starting place would be to count the number of paths to every intersections, starting from the top and only moving "downwards". Then, let's try to revisit our polynomial-multiplication example and figure out how the whole picture fits together!

Count the number of paths to each intersection, and record your findings in the space below. You'll have to figure out some sensible, systematic way to write down all these numbers!

Last, recall that the numbers in Pascal's Triangle appeared naturally when we looked at expansions of the polynomials $(a + b)^n$. Where could we possibly find the numbers you wrote down above in polynomial patterns, in a similar sense?

That concludes our second session! Make sure to check out the problem set for this week[4] and feel free to reach out to me (**mgusak@uwaterloo.ca**) with questions or cool math. See you next week!

[4] I worked very hard on it!